IRB Asset and Default Correlation: Rationale for the Macroprudential Add-ons to the Risk-Weights

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ABSTRACT

Basel III allows for the use of statistical models. It is called the internal-ratings-based (IRB) approach and is based on the (Vasicek, 2002) model. It assumes assets returns are standard normally distributed. It suggests incorporating different asset correlation (R) functions to assess credit risk for the loan portfolio, or the risk-weighted assets (RWA). The asset correlation function solely depends on the individual default probability (PD) given certain credit exposure type.

At the same time, the IRB approach requires developing PD models to predict the discrete default event occurrence. This means that the IRB approach is based on the Bernoulli trials. We investigate the impact of the asset returns' correlation for the Bernoulli trials. We show that when Bernoulli trials are considered, the credit risk estimation significantly deviate from the values derived under the normality assumption of asset returns.

We investigate the simulated and real-world credit rating agencies' data to specifically demonstrate the scale of the credit risk underestimation by the IRB approach. Therefore, macroprudential add-ons are of use to offset such IRB limitations.

Keywords: Basel II, IRB, correlated defaults, asset correlation, binomial distribution, Bernoulli trials, macroprudential add-ons.

JEL codes: C25, G21, G28, G32, G33.

1. INTRODUCTION

Capital adequacy ratio (CAR) became an internationally accepted indicator of the bank's financial standing in 1988 when the Basel I Accord was adopted by the Basel Committee on Banking Supervision (BCBS, 1988). CAR equals to the amount of the bank own funds (capital) divided by the amount of the risk-weighted assets (RWA). RWA is a product of a risk-weight (RW) and an asset (A) amount. Originally Basel I defined several categories of fixed risk-weights.

In 2006 Basel II allowed using bank own loan default statistics and mathematical models to estimate the risk-weights by banks themselves (BCBS, 2006). The approach was called an internal-ratings-based (IRB) one. By 2019 around two thousand of the world largest banks operated IRB (Ermolova & Penikas, 2019, p. 106). We recommend referring to (Penikas, 2015) for more details on the overall banking regulation evolution developed by the Basel Committee and to (Penikas, 2020a) for more details on the evolution of the IRB approach.

Concisely, the IRB approach is based on (Vasicek, 2002) loan portfolio model. It departs from the key assumption of the normally distributed asset returns. However, the Basel Committee requires predicting the discrete default events within the IRB approach. This means that there is a change in (Vasicek, 2002) key model assumption. That is why it is important to investigate how the transit from the Gaussian asset returns to the Bernoulli random variables standing for default events may change the credit risk assessment. In case the deviations are material, there comes a need to offset it in the day-to-day supervisory practice of the regulator. Macroprudential add-ons (mark-up) to the IRB risk-weights are of value here. Thus, the objective of the current working paper is to determine the values and drivers for such macroprudential add-ons given the difference in IRB assumptions from the banking practice.

There are two primarily findings. First, we show that the values of the discrete default events' correlation varies because of the DR variances given the same PD. The IRB approach does not account for the DR variance as a driver when computing the asset correlations. Second, the offered parametrization for the default correlation enables to recalibrate the AC function with the recent data available to the credit institution, i.e. to have both IRB inputs up to date (PD and AC). Such an update may signal whether the regulator has to introduce non-zero macroprudential add-ons to the risk-weighted assets or not.

To guide the reader on how we got the AC and DC function parametrization and the values of the macroprudential add-ons, the paper has the following structure. Section 2 presents the literature overview. It consists of two parts. Part 1 briefly shows the Basel IRB framework that takes the asset correlation as an input. Asset and default correlation definitions are discussed there. Part 2 discusses how various researchers handled the issue of the asset and default correlation evaluation assuming the Gaussian asset returns. Part 3 briefly refers to macroprudential policy tools. Those are used to compensate for the shortcomings of the microprudential regulation, including the IRB one. Section 3 proves the suggested parametrization of the asset and default correlation for the discrete default events. Section 4 runs the robustness check for the suggested parametrization using simulated data with negative, neutral and positive correlation of discrete defaults. Section 5 presents evidence from the publicly available data to demonstrate the impact of the suggested parametrization. Particularly, we were able to find out the situations when the IRB approach and 2003-04 AC function assuming correlated the Gaussian asset returns ca. twice underestimate the credit risk compared to the approach assuming the correlated discrete defaults. Section 6 concludes underlining the importance of such an underestimation treatment via macroprudential add-ons to the IRB risk-weights.

2. LITERATURE REVIEW

PART 1. BASEL IRB FRAMEWORK

(Vasicek, 2002) model underlies the IRB approach in Basel II and III accords (BCBS, 2005c). (Merton, 1974) model forms the basis for the Vasicek one. Thus, Merton defined default for the company as a situation when the market value of its publicly traded assets goes below the amount of its debt. Vasicek used Merton definition and expanded it by saying that the market value of the i-th company asset, or its asset (value) return, (Z_i) depends on the two factors: systemic (X) and idiosyncratic (ε_i) as follows in (1).

(1) $$Z_i = X\sqrt{R} + \varepsilon_i\sqrt{1-R},$$

where $Z_i \sim N(0;1)$ - asset (value) return of the i-th borrower, $X \sim N(0;1)$ - systemic factor, $\varepsilon_i \sim N(0;1)$ - idiosyncratic factor, R - dependence parameter, $N(0;1)$ - standard normal distribution.

(Vasicek, 2002) assumes that the systemic factor contributes to the asset market value with a share of square root of R. As a note for future discussion, he does neither introduce default correlation, nor suggests any parametrization for R. However, he notes, if there was no R, the resultant default rate distribution would converge to a Gaussian one.

Terms ‘asset and default correlations’ appear in the working paper No. 14 (BCBS, 2005). Specifically, R is called asset correlation. We will cover the ‘default correlation’ definition from (BCBS, 2005) later in Part 2 of the literature review. However, the working paper (BCBS, 2005) has a different expression for the asset value Z_i that is conventional to the probability theory:

(2) $$\mathrm{Z_i = rX + \varepsilon_i\sqrt{1-r^2}}.$$

Formula (2) is conventional to the generation of a normally distributed random variable Z that co-depends with another normally distributed random variable X with the correlation coefficient (r). Formula (1) is related to formula (2) via the following relationship:

(3) $$\mathrm{R} = \mathrm{r}^2.$$

Such a fact does not imply that the asset or default correlation becomes neither significant, nor insignificant. As we show later in (23), the higher the variance of the default rate is, the higher the default correlation r becomes. As the Figure 3 below shows, the closer the default rate gets to zero or one, the larger the asset correlation R gets. Thus r stands for the mathematical correlation (let us refer to it as default correlation, DC, as (Foulcher, Gouriéroux, & Tiomo, 2005) also do), whereas R stands for the IRB asset correlation (AC). By introducing such a definition, asset correlation R is always positive. However, the default correlation might still be negative. From the credit risk management perspective formula (1) does not allow to account for the risk diversification when there is a negative correlation of the credit risk events. Formula (2) does allow for the risk diversification, as we show below, specifically in case of a finite and not large number (ca. one thousand) of observations (the total number of borrowers or credit facilities). (Gordy M. , 2000, pp. 147, eq. C.1) and (Gordy & Heitfield, 2010, pp. 46, eq. 3) also use formula (2) allowing for negative correlation coefficient (r).

However, BCBS decides to proceed with the formula (1). Formula (1) serves the baseline for the following ultimate IRB risk-weighting formula (BCBS, 2005c) of Basel II and then Basel III. Without loss of generality, we demonstrate formula (4) without subtraction of the expected loss part (-PD) and without multiplication by loss given default (LGD) and exposure at default (EAD). $\mathrm{RW_i}$ in the presented form is the total capital requirement for the credit risk. Regulator requires decomposing it in two parts: expected loss (EL) and unexpected loss (UL). EL goes into the numerator of the CAR formula; UL – to CAR denominator. Later we will show policy implications with respect to the total capital requirement. That is why it is sufficient to proceed with formula (4). It assumes that there is an infinite number of infinitely small identical exposures, i.e. the portfolio is perfectly granular.

(4) $$RW_i = N\left(\frac{N^{-1}(PD_i)+N^{-1}(0.999)\sqrt{R}}{\sqrt{1-R}}\right),$$

where RW_i – capital requirement for i-th borrower or facility; acts as a multiplier for assets within the CAR denominator;

0.999 - confidence level for systemic risk-factor (X) realization;

$N(\quad)$ – standard normal probability distribution function; $N^{-1}(\quad)$ – its inverse;

PD - probability of default lying in the range of [0; 1], its stands for the idiosyncratic risk-factor value.

The AC function reflects the degree of borrowers' co- or counter-dependence. We may deem it to be a regulatory novelty as Vasicek did not suggest any functional dependence of R and PD. This is why our further modification of R does not make it impossible to apply the general Vasicek model. The IRB AC function is as follows (see par. 53, 54, 64, 118, 120, pp. 63-75 of (BCBS, 2017)).

(5) $$R_i(PD_i) = R_0 + R_{MIN} \cdot \mathrm{Q}(PD_i, \mathrm{coef}) + R_{MAX}\big(1 - \mathrm{Q}(PD_i, \mathrm{coef})\big),$$

where $\mathrm{Q}(PD_i, \mathrm{coef})$ - weight as a function of PD and calibration coefficient (coef) in the following form

(6) $$\mathrm{Q}(PD_i, \mathrm{coef}) = \frac{1-e^{-coef \cdot PD_i}}{1-e^{-coef}},$$

where R_... – the asset correlation; R_0 – the fixed value if R is independent of PD; R_{MIN} – minimum value of the asset correlation attained at the largest PD; R_{MAX} – maximum value of asset correlation attained at the smallest PD. Please, refer to Annex 1 in (Penikas, 2020a) for the parameter values per the credit exposure types.

As we can see from (5), R depends only on PD. BCBS wanted to reflect the following idea. The higher PD is, the lower the R value is. This means, the less creditworthy the borrower is, the less its credit risk is augmented when crisis comes in all other things being equal. The reason for this is that one's risk assessment is already relatively high. Figure 1 has the visual representation of various R functions.

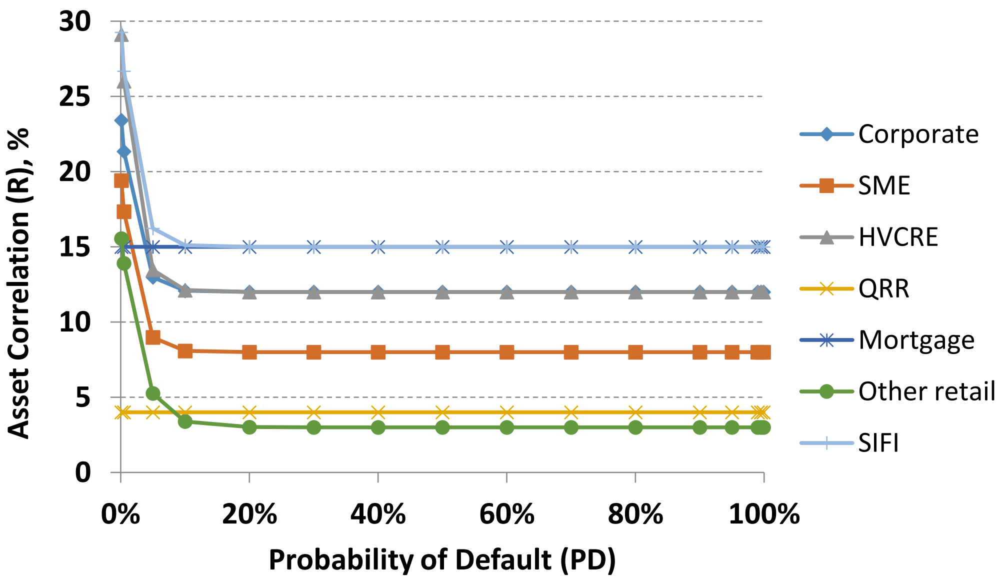


Figure 1. Asset correlation per credit product types according to Basel II (III).

Basel working paper 14 (BCBS, 2005, p. 51) postulates that R values were calibrated in 2003 using quantitative impact study (QIS) 3.[1] This raises an issue whether the preserved R functions' calibration is representative in 2020 and, if it is not, then how we should recalibrate it; or introduce the macroprudential add-ons to the risk-weights to adjust for the material discrepancies that may come out.

The key implication of the AC functions variation by product types is the fact that capital requirements differ given the same default statistics. Annex 2 offers two examples (see row 7): for mean DR equal to 5% and 10%. For instance, for the mean DR of 5% and DR variance of 1% the capital requirement (RW_i) – i.e. the proxy for the mean DR and a bonus for asset (default) correlation – varies from the lowest of 14.73% of the credit exposure amount if it is qualifying revolving retail (QRR) loan to 33.11% if it is a loan to a SIFI. We may compare both figures as they are ultimately summed up when computing total RWA.

To sum up, the IRB framework does not allow for the negative default dependencies. The asset correlation function's parametrization may be outdated and may require validation and recalibration or the macroprudential add-ons to the risk-weights introduction.

PART 2. APPROACHES TO ASSET AND DEFAULT CORRELATION MODELING

There are several milestones in the literature when we trace asset and default correlation modeling. To list those chronologically, they are (Vasicek, 2002), (Duffie & Singleton, 1999), (Li D. , 2000), (Nagpal & Bahar, 2001), (Lopez, 2002), (Duffie & Singleton, 2003), (BCBS, 2005), (Blochwitz, Martin, & Wehn, 2006), (Patel & Pereira, 2008), (Ozdemir & Miu, 2009), (Duellmann, Kull, & Kunisch, 2010), (Vozzella & Gabbi, 2010), (Li, Shang, & Su, 2015), (Khorasgani & Gupta, 2017), (Wunderer, 2019), (Baldwin, Alhalboni, & Helmi, 2019).

(Duffie & Singleton, 1999) are the first authors to mention the correlation of defaults as a formal term. They discuss the importance of such a feature for the credit portfolio management by showing the increased DR variance in its presence. However, they do not attempt calibrating it to historical data. Later (Li D. , 2000) and (Duffie & Singleton, 2003) discuss how copulas might be applicable for the joint default dependence simulation. Though both use the Gaussian copula, (Li D. , 2000) merely shows the methodology of how to apply it, whereas (Duffie & Singleton, 2003) claim that the Gaussian copula is inadequate in fitting the data.

(BCBS, 2005, p. 51) suggests modification of a PD model accuracy validation procedure (i.e. the binomial test) given the presence of the asset correlation. It still warns that direct use of recommended asset values from QIS 3 (please, see Annex 1 in (Penikas, 2020a, p. 98)) would significantly extend the confidence interval for the binomial test. This implies making

[1] BCBS distributed QIS 3 data collection template in November 2002 and collected it in 2003. URL: https://www.bis.org/bcbs/qis/qis3.htm [accessed: December 13, 2019].

conclusions that the model is accurate more often than it is really needed or expected by someone. As a sort of response to (BCBS, 2005), (Blochwitz, Martin, & Wehn, 2006, p. 295) suggest using default, not asset correlation in the Basel IRB risk weighting formula. (Blochwitz, Martin, & Wehn, 2006, p. 295) also note that in practice default correlation may be up to 3%. This coincides with the findings for the French data estimates by (Foulcher, Gouriéroux, & Tiomo, 2005) and for the German companies by (Hamerle & Rosch, 2006, p. 21). Both values are much lower than the values from the Basel AC function resulting in a compromise, i.e. the confidence interval for the portfolio PD forecast becomes larger than in the case of the default independence, though smaller than if validators used Basel II AC values. As a result, validators deem models to be accurate less often than if when they used the IRB asset correlation R values.

(Lopez, 2002) and (Patel & Pereira, 2008) try to investigate the drivers of asset and default correlation. (Lopez, 2002) runs a more descriptive analysis, whereas (Patel & Pereira, 2008) undertake a regression one. However, (Lopez, 2002) obtains asset correlation values that are comparable to the regulatory one. (Patel & Pereira, 2008) use a proxy for default correlation from the copula dependence parameters. The latter ones are computed for the stock asset returns during five years of data. Thus, the major shortcoming of (Patel & Pereira, 2008) work is that they do neither demonstrate the distribution of the default rates, nor the derived default correlations, nor their proxies, nor run a goodness-of-forecast testing for the obtained proxy default correlation values.

Let us review the approach to the default correlation (DC) definition more closely. (Nagpal & Bahar, 2001) offer DC parametrization of the type for two credit risk categories (ratings) i and j:

(7) $$DC = \frac{PD(Y_i,Y_j)-PD(Y_i)\cdot PD(Y_j)}{\sqrt{PD(Y_i)(1-PD(Y_i))}\cdot\sqrt{PD(Y_j)\left(1-PD(Y_j)\right)}} = \frac{PD(Y_i,Y_j)-PD^2}{PD\cdot(1-PD)},$$

where $PD(Y_i,Y_j)$ is the joint default probability for the two identical random variables (variates) Y_i and Y_j of a following type:

(8) $$Y_i = \begin{cases} 0, \text{ with probability } (1-PD) \\ 1 \text{ (default or } Z_i < r_{Di}), \text{ with probability } PD' \end{cases}$$

where $Y_i = Y_j = 1$ is the default event occurrence for a single borrower (facility) when the asset (value) is less than the interest rate on its debt, i.e. $Z_i < r_{Di}$; PD is the marginal (individual) default probability.

(BCBS, 2005, p. 48) offers a default correlation (DC) formula assuming the correlated Gaussian asset returns. Author was unable to find the publicly available first version of the mentioned guidance if not to consider a page within (BCBS, 2000, pp. 38-39). (Aramonte & Avalos,

2020) seem to also use such an approach though they do not disclose granular approach to computation.

(9) $$DC = Corr\left(Y_i, Y_j\right) = \frac{N(\gamma,\gamma)-PD^2}{PD\cdot(1-PD)},$$

where $N(\gamma, \gamma)$ is a bivariate standard normal cumulative density function for the two random variates Y_i and Y_j, such that $E(Y_i) = E\left(Y_j\right) = F_Y(\gamma) = PD$ or $\gamma = N^{-1}(PD)$, $N^{-1}(\quad)$ is the inverse standard normal probability distribution function; and asset correlation is $R = Corr\left(Z_i; Z_j\right)$. Illustrative values of $N(\gamma, \gamma, R)$ are available in Annex 4. Default correlation may be expected to be low when one considers underlying Gaussian asset value returns (Fantazzini, 2009, p. 119). However, even for Gaussian random variates the shadow area from Figure 2 in (Fantazzini, 2009, p. 119) strongly depends upon the chosen threshold (i.e. debt levels r_{Di} from the Vasicek model).

When one compares (7) to (9), one may conclude that (BCBS, 2005, p. 48) assumes the following (Gordy M. , 2000, pp. 148, eq. C.4):

(10) $$P\left(Y_i, Y_j\right) = N(\gamma, \gamma).$$

Then $N(\gamma, \gamma)$ resembles the concept of the upper (lower) tail dependence index for the joint probability distribution (see (Nelsen, 1999), for instance). We wish to highlight once again that such a transition in (10) is justified only for correlated Gaussian asset returns ((Foulcher, Gouriéroux, & Tiomo, 2005, pp. 5, formulas (2.5), (2.6)), (Gordy & Heitfield, 2010, p. 47), (Wunderer, 2019)).

For the future discussion, let us remember the modification coming from the (BCBS, 2005, p. 48) formula (9) using (Vasicek, 2002) assumptions:

(11) $$N(\gamma, \gamma) - PD^2 = DC \cdot PD \cdot (1 - PD).$$

(Duellmann, Kull, & Kunisch, 2010) find that asset correlation values differ when one derives it either from the asset price (quote) data or from the default time series. (Wunderer, 2019, pp. 4, eq. 2.4) tries to explain this observation. Though (Wunderer, 2019) starts with the discrete defaults, he uses (10) for the Gaussian random variates. (Wunderer, 2019) supposes that the portfolio inhomogeneity is the cause of the differences in the correlation estimates based on the asset return and the default statistics. To prove it, he departs from the following conventional theoretical Bernoulli random variate variance definition:

(12) $$Var(Y_i) = PD \cdot (1 - PD),$$

and DR definition:

(13) $$DR = \frac{\sum_{i=1}^{n} Y_i}{n},$$

to come to the following DR variance as first shown in (Gordy & Heitfield, 2010, pp. 58, eq. 20):

$$Var(DR) = N(\gamma,\gamma) - PD^2 + \frac{PD - N(\gamma,\gamma)}{n}. \quad (14)$$

He does not write this, but if we let n go to infinity, we obtain exactly the same component that (BCBS, 2005) presented, i.e.

$$Var(DR) = N(\gamma,\gamma) - PD^2 + \frac{PD - N(\gamma,\gamma,R)}{n} \xrightarrow{n\to\infty} N(\gamma,\gamma) - PD^2. \quad (15)$$

If one joins formula (11) and (15), then one can obtain that

$$DC = \frac{Var(DR)}{PD \cdot (1 - PD)}. \quad (16)$$

As we show later, formula (16) is also the needed parametrization for the correlation of discrete defaults (not Gaussian asset returns) when the number of borrowers goes to infinity.

To sum up, we have shown that the existing literature already provides the default correlation parametrization for the discrete default events in (16). However, while (BCBS, 2019) requires predicting discrete defaults, it still uses an assumption of the correlated Gaussian asset returns in the Vasicek set-up of the IRB approach. The working paper by (Khorasgani & Gupta, 2017) also stated that the IRB asset correlation was wrongly parametrized.

As a robustness check for the formula (16), we derive it separately and model several distributions with the various marginal default probabilities and the various default correlation parameters. Then we compare the suggested parametrization with the parameter estimates when the data generation process is known. As an alternative, we benchmark our results to that of (Wunderer, 2019). We show that (Wunderer, 2019) overestimates asset correlation (R) and capital requirements for more correlated and more risky borrowers if those related to the discrete default events (not to the continuous Gaussian asset returns).

PART 3. MACROPRUDENTIAL POLICY MEASURES.

We additionally investigate the real-world credit rating agencies' data to demonstrate the scale of the credit risk deviation. In cases where there is a substantial underestimation according to the IRB approach, the macroprudential measures are useful to offset for those. The macroprudential tools were discussed since 1970s in the auspices of the BCBS (Clement, 2010) and since 2000 publicly (Crockett, 2000), (Borio, 2003). The extensive reviews of the macroprudential policies are available in (Schoenmaker, 2014), (Kahou & Lehar, 2017), (Lubis, Alexiou, & Nellis, 2019). (Penikas, 2020a) claims that the macroprudential add-ons to the risk-weights (or increasing the risk-weights) should be preferred to mere change in the overall minimal capital adequacy ratio (CAR) requirement from the bank stakeholders' perspective. The latter change in the CAR minimum impacts only those banks that are close to the threshold. Therefore, such a change may trigger no information signal (CAR value does not change) for the banks with the different risk profiles, but that end up above the new CAR threshold. However, the former macroprudential measure of a change in the risk-weights creates signals for all the banks as it implies change in the CAR. Such a change is more pronounced for the banks that are more exposed to the lending segment targeted by the macroprudential policy.

Thus, the macroprudential policy tools including add-ons to risk-weights form an integral part of the modern prudential banking regulation and supervision. Their efficiency is regularly discussed and assessed. For instance, (BIS, 2020) reviews such tools implementation in the Asian countries (Australia, Indonesia, New Zealand, Philippines, Thailand). Latin American countries (Argentina, Brazil, Colombia, Mexico and Peru) are considered by (Gambacorta & Murcia, 2020); Turkey is handled by (Yarba & Güner, 2020); Southern Korea is the focus of (Kim & Oh, 2020); European Union-wide effects are surveyed by (Budnik & Jasova, 2018) and (Dautović, 2019); the Great Britain and Ireland are investigated by (Basten, 2020) and (McCann & O'Toole, 2019). Macroprudential policy measures application for Russia is covered by (Danilova & Morozov, 2017), (Ivanova, Andreev, Sinyakov, & Shevchuk, 2019), (Khotulev & Styrin, 2020).

3. THEORETICAL PART

Our Assumptions

Here below we list the assumptions of the further research:

(1) We investigate solely the impact produced by a single parameter – the probability of default (PD). We do not consider the impact of other credit risk drivers, including LGD, EAD, M (maturity). Such an investigation may be a focus of a separate paper. By doing this, we follow (Vasicek, 2002) as he also assumed “gross loss (before recoveries)”. Though we investigate PD and DR only, this working paper findings are applicable to both the Foundation and Advanced IRB approaches as both depend upon a PD.

(2) We deal with a single PD model segment. If there is a single PD model for a bank portfolio (e.g. for a narrowly focused business segment or a monoliner bank), than the working paper findings are applicable to the entire portfolio.

(3) This model segment has a uniform distribution of borrowers by PD.

(4) Basel IRB approach requests methodologically differentiating approaches to PD model development for corporate and retail claims. For the corporate claims a PD model should be build at the borrower level. For the retail claims it should be built at the facility level. Disregarding the differences in such a methodological treatment of the PD modeling, ultimately one sums up the risk-weighted assets for the corporate and retail borrowers. This means that our research may compare results for the different asset classes.

(5) We assume that the PD used for the default correlation derivation reflects the long-run default experience of a bank for the lending segment of interest.

Default Correlation Parametrization Derivation

Consider the variance of the two identical Bernoulli random variates Y_{i} and Y_{j} introduced in (8) (Kelbert & Sukhov, 2010, p. 78) or (Van Der Geest, 2005, p. 145):

$$Var(Y_{\mathrm{i}} + Y_{\mathrm{j}}) = Var(Y_{\mathrm{i}}) + Var(Y_{\mathrm{j}}) + 2 \cdot Cov(Y_{\mathrm{i}}; Y_{\mathrm{j}}). \tag{17}$$

If r is a correlation coefficient between identical Y_i and Y_j and $Var(Y_i) = Var(Y_j)$, then their covariance equals to:

$$Cov(Y_i; Y_j) = r \cdot \sqrt{Var(Y_i)} \cdot \sqrt{Var(Y_j)} = r \cdot PD \cdot (1 - PD). \tag{18}$$

We will call r a **default correlation** as it estimates the correlation between Bernoulli trials that indicate default. The variance of a sum of n Bernoulli random variates equals to:

$$Var(n \cdot Y_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(Y_i; Y_j) = n \cdot Var(Y_i) + n(n-1)Cov(Y_i; Y_j), \tag{19}$$

where for a single Bernoulli random variate Y_i, the variance is mentioned in formula (12). Putting (12) and (18) into (19) yields us:

$$Var(n \cdot Y_i) = n \cdot PD \cdot (1 - PD) + n(n-1) \cdot r \cdot PD \cdot (1 - PD). \tag{20}$$

In case of the Bernoulli trials' independence we simply have $Var(n \cdot Y_i) = n \cdot PD \cdot (1 - PD)$ (Kelbert & Sukhov, 2010, p. 83). (Witt, 2014, p. 4268) departs from the same standing point to derive the formula for the distribution of the correlated Bernoulli trials, or the absolute number of defaults. However, he does not provide an explicit parametrization for the default correlation function for the case of the discrete default events. As a consequence, he does not evaluate impact on the capital requirements. We do this in section 5 of the current working paper.

Consider the default rate (DR) as in formula (13). To estimate the credit risk at the portfolio level, we substitute PD with its sample estimate in (20), i.e. with the mean value of DR ($PD = \overline{DR}$). Then the DR variance is:

$$Var(DR) = \frac{Var(n \cdot Y_i)}{n^2} = \frac{1}{n} \cdot \overline{DR} \cdot (1 - \overline{DR}) \cdot (1 + (n-1) \cdot r). \tag{21}$$

From here we may derive the correlation value floor, that was noted by (Preisser & Qaqish, 2014):

$$r \geq -\frac{1}{n-1} \text{ or } r \geq 0 \text{ for } n \to \infty. \tag{22}$$

This means that when n goes to infinity, one cannot expect a negative default correlation as a single parameter over a credit portfolio. However, even for a thousand borrowers a minor negative correlation is feasible. Let us derive the default correlation value (r) from (21). We have:

$$r = \left(\frac{n}{n-1}\right)\left(\frac{Var(DR)}{\overline{DR}\cdot(1-\overline{DR})}\right) - \left(\frac{1}{n-1}\right). \quad (23)$$

When one wishes to obtain asset correlation R, it is sufficient to take the power of two for r as done in (3). Jointly formulas (3) and (23) for the correlated discrete defaults substitute formulas (5), (6) for the correlated continuous Gaussian asset returns. If we simplify (23) by taking the limit for n going to infinity, then the proxy for the default correlation is:

$$r = \frac{Var(DR)}{\overline{DR}\cdot(1-\overline{DR})}. \quad (24)$$

In formula (24) we obtain the result of formula (16) that is a combination of (BCBS, 2005) and (Wunderer, 2019). It is interesting to mention here the survey by (Wehrspohn, 2004). He argues that PD contributes more to the credit loss amount than the correlation. His statement may be correct for the correlated Gaussian asset returns similarly to (Fantazzini, 2009, p. 119). However, for the correlated Bernoulli trials both PD and r multiplicatively contribute to the default rate variance. The contribution depends upon their values.

For the regulatory purposes, we suggest the following modification to the Basel IRB risk-weighting formula instead of formula (4). It allows for the negative default correlation.

$$RW_i = N\left(\frac{N^{-1}(PD_i)+r\cdot N^{-1}(0.999)}{\sqrt{1-r^2}}\right), \quad (25)$$

where r comes from formula (23).

There may be counterarguments against allowing for the negative default correlation in (25). For instance, one may recall the lower boundary for the correlation value from (22), i.e. for the large number of borrowers the correlation r is non-negative. We need to underline here that the presence of the negative default correlation is feasible in the case of *low default portfolios (LDP)*, i.e. when the number of borrowers is limited (BCBS, 2005b), (Benjamin, Cathcart, & Ryan, 2006). As an illustration, one may imagine an IRB segment of lending to one hundred large borrowers. Then a situation with $r \approx -1\%$ is feasible. For comparison, (Penikas H. , 2020b, p. 124) separately discusses that the LDP in itself may be an outcome of +100% default correlation.

Let us now proceed with the robustness check of the proposed parametrization for the default correlation r.

4. ROBUSTNESS CHECK: SIMULATED DATA

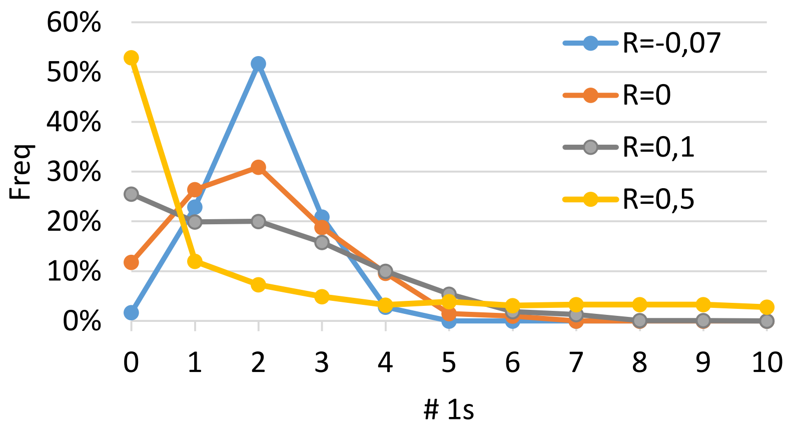
To obtain the correlated Bernoulli random variate probability density distributions, we use a genetic algorithm described by (Kruppa, Lepenies, & Jung, 2018). There are alternative approaches for the correlated Bernoulli random variates' distribution generation. Those include (Lunn & Davies, 1998), (Zaigraev & Kaniovski, 2013), (Preisser & Qaqish, 2014), (Witt, 2014), (Bakbergenuly, Kulinskaya, & Morgenthaler, 2016), (Chiu, Jackson, & Kreinin, 2017). (Kruppa, Lepenies, & Jung, 2018) allow getting the distributions of the negatively correlated Bernoulli outcomes. So we proceed with it.

(Kruppa, Lepenies, & Jung, 2018) algorithm has two steps. First, we generate a vector with the required marginals, i.e. with the needed number of zeros (non-defaults) and ones (defaults) per columns. A column stand for the hypothetical borrower default status at the different hypothetical points in time (rows). For simplicity we take ten variates ($n = 10$) and the number of rows (observations) equals to 100. Second, one starts randomly perturbing (exchanging places) within any column so that marginals (mean PDs) do not change, but the default correlation changes. We introduce an error as the square root of the sum of the element-wise differences of the two correlation matrices: the one under perturbation and the target one. The target correlation matrix has r values everywhere except the principal diagonal. For the simplicity, we consider the matrix differences only for its half, i.e. for the below diagonal elements only. The optimization procedure is iterative and may not always converge, as (Kruppa, Lepenies, & Jung, 2018) note. That is why we set limits on the acceptable error (10%) and the maximum number of iterations (10 thousand). We report both in Annex 1 for the found solution (see columns 7 and 6, respectively).

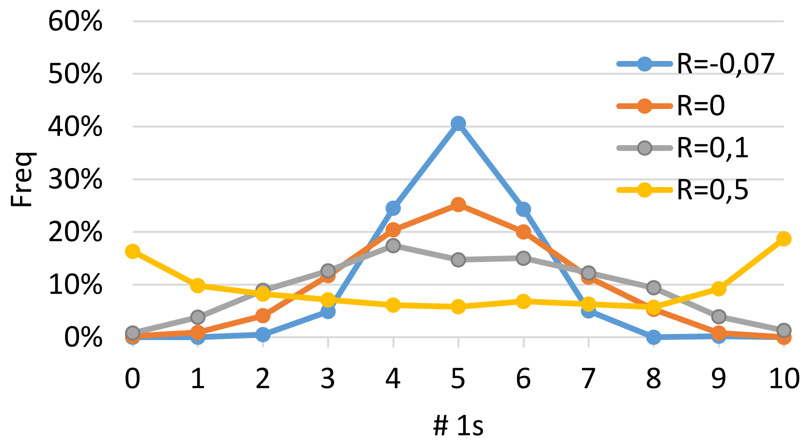
For the simulation purpose, we choose four values of the marginal PD (5, 10, 25, 50%) and six values of the default correlation r (-10, -5, 0, +5, +10, +20). For each parameter set we generate a new initial DR distribution. That is why there may be statistically insignificant differences in the mean DR values given the same PD input in Annex 1 (compare columns 8 and 3, respectively).

One may notice from Figure 2, that for the significantly positive default correlations the DR probability density distribution becomes wider and bimodal. (Witt, 2014, p. 4273) is the first to show such a bimodality feature. Then (Li, Shang, & Su, 2015) independently notice the same fact when they observe that the frequency of the high and low risk categories (credit ratings) is higher, than that of the mid-risk categories.

When the default correlation is negative, the DR distribution becomes narrow and goes to a degenerate state to concentrate around the mean PD value. (Witt, 2014) does neither demonstrate this, nor consider in his examples, though he accepts that negative correlation of Bernoulli trials is feasible. In contrast to (Witt, 2014), (Van Der Geest, 2005) deals with the negative correlation for the illustrative purposes. The illustration from the Figure 2 supports the statements by (Preisser & Qaqish, 2014). They say that the variance of the correlated random variate distributions decreases for the low (negative) correlation and increases – for the high (positive) one.



(a) PD = 20%



(b) PD = 50%

Figure 2. Illustration of DR distributions for various PD and correlations of Bernoulli random variates.

Note: # 1s – the number of defaults (ones).

Compare Figure 2 to the statement by (Ozdemir & Miu, 2009, pp. 70-71, Fig. 2.4). They claim that through-the-cycle (TTC) rating philosophy implies wider DR distribution, whereas the point-in-time (PIT) rating philosophy leads to a narrow DR distribution. This data simulation exercise vividly illustrates that (Ozdemir & Miu, 2009) claim is correct if and only if they associate TTC rating philosophy with the higher default correlation and PIT one – with lower one. However, we would expect an opposite, i.e. the DR distribution for the PIT DR is wider than for the TTC one because of the volatility coming from the economic cycle stages.

Figure 3 depicts the dependence of the asset correlation values against the marginal PD ones. Several lines correspond to the different values of DR variance. We advise the reader to compare Figure 3 to Figure 1. As for the suggested approach illustration in Figure 3, we present values from formula (3), i.e. the squared values (r^2) for the visual comparability with Figure 1.

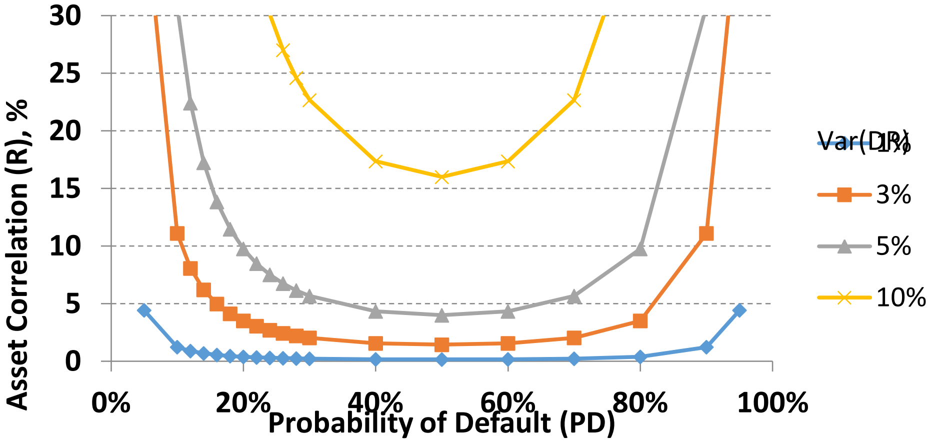
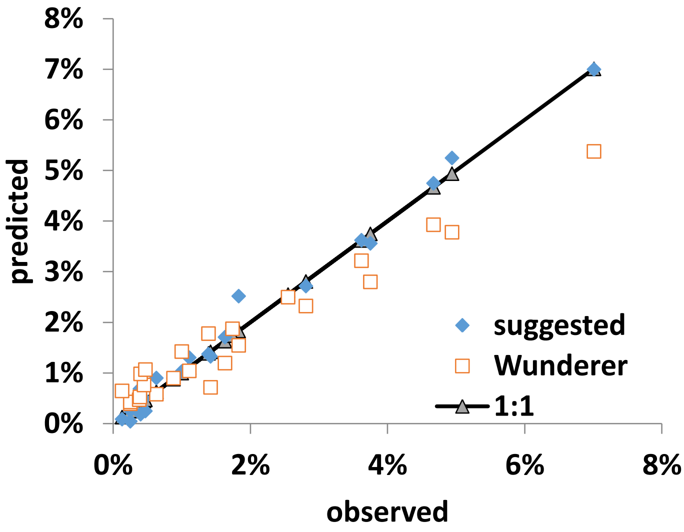


Figure 3. Illustration of the Suggested Asset Correlation (R) parametrization and its dependence with PD.

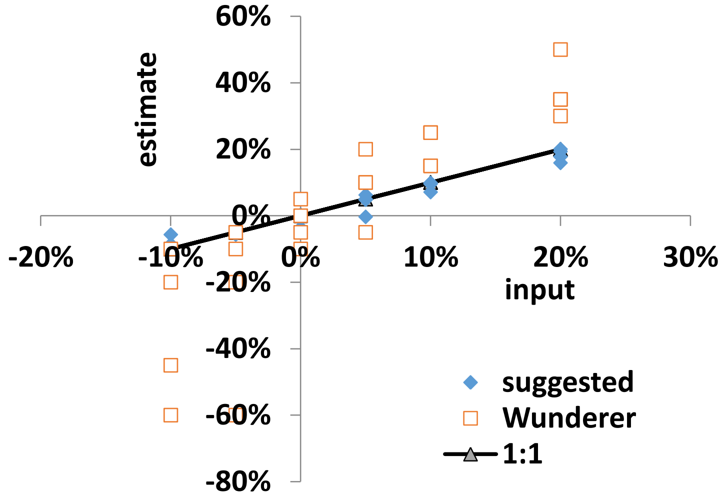
One may see that the regulatory asset correlation R formula (5), (6) in Figure 1 generally reflects the concept of a declining asset correlation value for the higher PD values when PD is less than 50%. However, the correlated Bernoulli random variates do not demonstrate the patterns used in regulatory regulation for PD in excess of 50%. Then AC starts upward sloping (see Figure 3). Such a positive dependence of asset correlation and default probability was underlined by (Vozzella & Gabbi, 2010). Second, AC for the correlated discrete defaults varies because of the changes in the DR variance. It cannot vary because of a credit product type given the same PD as in Figure 1 unless this credit product is characterized by a different DR variance. This means that the asset correlation can be uniquely statistically calibrated using the data on the mean default rate and its variance.

As you can see from Figure 4, the suggested approach in formula (21) mostly ideally predicts the DR variance values and quite closely fits the input correlation r value. The minor discrepancies result from the fact that the optimized (iterated) correlation matrix for the generated joint distribution of Bernoulli variables does not always converge to the target correlation matrix. One may note a high value of error, in excess of its limits when the algorithm reached the maximum number of iterations, see column 7 in Annex 1.

We consider (Wunderer, 2019) as the most recent benchmark research for AC and DC parametrization in Figure 4. He assumes the underlying correlated Gaussian asset returns when estimating AC. See Annex 1, columns 9-11 for the DR variance comparison and columns 14-16 for the default correlation r comparison.



(a) DR Variance



(b) Correlation (r)

Figure 4. Comparison of suggested approach and that of (Wunderer, 2019) based on simulated data with known characteristics.

The findings are listed below:

1. (Wunderer, 2019) overestimates the DR variance for the low (including negative) default correlation and/or low PD. From the risk-management perspective that may be fine as the approach produces the more conservative estimate of risk (proxied by DR variance, for instance) than it really is;
2. (Wunderer, 2019) underestimates the DR variance for the high (positive) correlation and/or high PD. This is not acceptable for the risk-management as the risk is underestimated compared to its true value. Here the macroprudential add-ons should be used to offset the credit risk underestimation.

3. As a consequence of the two above points, (Wunderer, 2019) overestimates the default correlation r for the high PDs.

We conclude that (Wunderer, 2019) approach inadequately assesses the credit risk when one has to deal with the correlated discrete default events. The reason is that (Wunderer, 2019) assumes correlated Gaussian asset returns for the asset correlation parametrization.

5. APPLICATION TO REAL-WORLD: REGULATORY IMPACT

We use the publicly available data from the international credit rating agencies of (Moody's, 2018, p. 27) and (S&P Global Ratings, 2019, p. 3). We compare the suggested approach in (24) assuming the correlated discrete defaults and the capital requirements assuming the correlated Gaussian asset returns. We specifically estimate the mean DR and DR variance on the two subsamples:

- till 2003 inclusive (as if it had formed the basis for the (BCBS, 2004) AC functions' calibration);
- till 2018 (as if we updated the calibration using the recent data).

As the dominant part of the international credit rating agencies' clients' pool consists of the non-retail (corporate) exposures, we assign the regulatory corporate exposure asset class when evaluating capital requirements for them. Annex 3 provides tabulated results for the time period covered by the data and the default rate mean and variance.

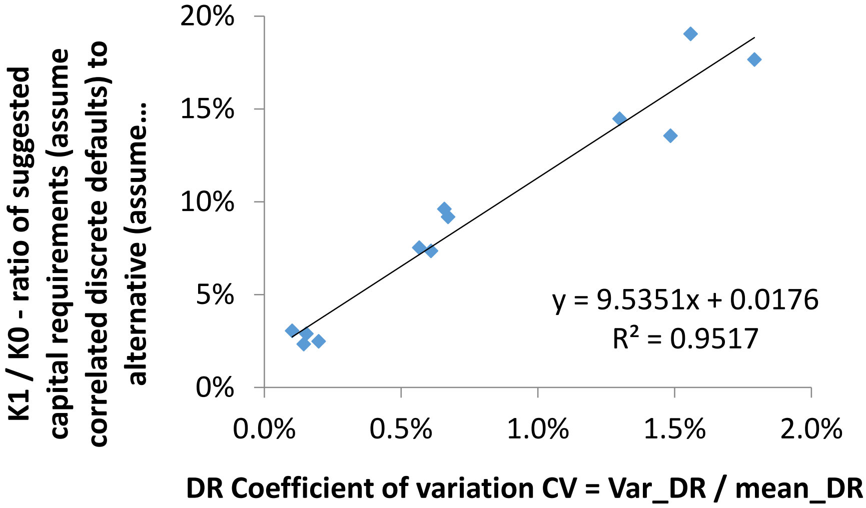


Figure 5. Comparison of suggested and capital requirements with respect to the coefficient of variation (CV) for the default rate.

Note: Horizontal axis has the coefficient of variance (CV). CV is the ratio of variance to the mean value. (Miller, 1991) discusses its properties.

We find that the higher the default rate coefficient of variance (CV) is for the empirical data under investigation, the closer the capital requirements from the suggested approach (cf. (23) and (25)) assuming the correlated discrete defaults are to the IRB ones (see (4), (5), (6)) assuming correlated Gaussian asset returns. However, even in the extreme case, the latter are still 5 to 50 times higher than that originating from the suggested approach (see row 16 in Annex 3). For instance, take S&P speculative grade (SG) borrowers' default statistics for 1981-2018. The mean DR is 3.9%, the DR variance equals to 0.07%. This implies the IRB capital requirement of 25.56% of the credit exposure (see row 10 of Annex 3) when assuming the correlated Gaussian asset returns. If we consider the correlated discrete defaults (as the data actually is), it is 4.51% (see row 14 of Annex 3). Thus, the ratio of the latter to the former equals to 4.51% / 25.56% = 17.7 %, i.e. the former approach assuming correlated Gaussian asset returns five times overestimate losses than the latter that assumes correlated discrete defaults.

The capital requirements under the assumption of the correlated Gaussian asset returns do not sufficiently well capture the discrete default data evolution from 2003 onwards. From one side, for the speculative grades (SG) we observe a decrease in the mean DR and DR variance disregarding the fact of the 2007-09 world financial crisis presence in the dataset. The IRB capital requirements for the correlated Gaussian asset returns do not fall as sharply as they do for the correlated discrete defaults (compare rows 11 to 15 in Annex 3). We may accept this from the risk-management point of view as the bank capital does not rapidly decrease either. However, from another side, for the investment graded (IG) borrowers we observe an increase in the mean DR and DR variance (e.g. for the Moody's data). The IRB capital requirements assuming the Gaussian asset returns in such a situation do not rise with the same pace (+20.5% from 2003 to 2018) as they do for the correlated discrete defaults (+28.2%). In contrast with the situation of the speculative grades, this one is not acceptable from the risk-management perspective. Nevertheless, this may seem sufficient as the IRB approach assuming the correlated Gaussian asset returns overestimates the credit risk amount in the absolute terms ca. 50 times for the IG grades compared to the assumption of the correlated discrete defaults (see row 16 in Annex 3).

We wanted to verify whether the situation of the excessive IRB capital requirements assuming the correlated Gaussian asset returns takes place for any combination of the credit risk parameters (mean DR and DR variance) compared to the values obtained for the correlated discrete defaults. It turned out that this is not always true. For this purpose we ran the illustrative calculations of the capital requirements for the different credit product types for the two values of the mean DR (5% and 10%) and the two values of the DR variance (1% and 2%). Annex 2 presents the capital requirements in rows: No. 7 (for the regulatory approach assuming continuous correlated Gaussian asset returns in formulas 3-5) and No. 10 (for the suggested parametrization assuming correlated discrete defaults in formulas 22, 25). Row 11 of Annex 2 reports the ratio of the two capital requirements. When such a ratio exceeds 100%, we have the situation of the credit risk underestimation according to the IRB capital requirements assuming the continuous correlated Gaussian asset returns. So far we may see that when the DR coefficient of variance (CV) is 40%, the IRB approach assuming the correlated continuous Gaussian asset returns underestimates credit risk mostly twice compared to the one assuming the correlated discrete defaults. The areas of the largest underestimation are plastic cards (QRR) and other retail loans (see values of 239% and 209% in row 11 of Annex 2a, respectively). The revealed underestimation is substantial and requires implementation of the macroprudential add-ons to the IRB risk-weights.

Same time, if CV=20%, the IRB capital requirements assuming the correlated Gaussian asset returns overestimate the credit losses compared to the situation of the correlated discrete defaults. Thus, the one has to compare an IRB credit risk estimate and the outcome resulting from the actual discrete nature of the default events. When the credit risk underestimation is revealed, macroprudential add-ons to the IRB risk-weights should be introduced to offset it.

6. CONCLUSION

Basel III (BCBS, 2017) and Basel Framework (BCBS, 2019) have inherited the IRB approach from Basel II (BCBS, 2006). (Vasicek, 2002) model makes the foundation for the IRB approach. It assumes normality of the borrowers' asset returns. Vasicek assumes that returns are correlated, i.e. there is an asset correlation. By introducing a threshold (quantile) for the Gaussian distribution, he obtains Bernoulli trials. They reflect whether the default has occurred or not. Due to the asset correlation, Vasicek arrives at the non-Gaussian unimodal portfolio loss probability density distribution.

Same time the IRB approach requires predicting the discrete events of the default. That is why by construction the IRB approach deals with the Bernoulli trials. In such a case we may call the correlation between the Bernoulli trials a default correlation. The correlated Bernoulli trials were studied before (Witt, 2014; Zaigraev & Kaniovski, 2013; Chiu, Jackson, & Kreinin, 2017). The correlated Bernoulli trials have a bimodal default rate probability density distribution.

However, there was no an explicit parametrization for the default correlation assuming discrete defaults before now. We are the first to offer such a parametrization. We show the implications for the IRB credit risk assessment if the correlation of the discrete defaults was used instead of the Gaussian asset return correlations. When the default rate coefficient of variance is 20%, the IRB capital requirements based on the correlated Gaussian asset returns exceed the credit risk amount corresponding to correlated discrete defaults. When the default rate coefficient of variance equals to 40%, the IRB capital requirements corresponding to the correlated Gaussian asset returns are significantly lower than the amount standing for the correlated discrete defaults. To sum up, in case the regulator finds that IRB credit risk assessment is significantly underestimated compared to the approach suggested in the working paper, the introduction of the macroprudential add-ons to the risk-weights help to offset such an underestimation.

Notations used in Annex 1

$\Phi\mathbf{2} = N(\gamma, \gamma, R)$ - joint bivariate standard normal distribution function (see Annex 4);
iter., # - number of iterations used to obtain the output (limit of 10 thousand iterations was set);
error – attained square root of sum of differences for two correlation matrixes: target and actual;
AVar – analytically derived (predicted) variance;
indep – as if the correlation of Bernoulli variates equaled to nil;
column 14 equals to column 4; it is replicated for visible convenience to compare to columns 15 and 16.
Suggested – approach from the current paper;
Wunderer – approach proposed by (Wunderer, 2019);

Notations used in Annex 2, 3

CV – coefficient of variation, ratio of DR variance to its mean value.
K – capital requirement to cover credit risk, equals to risk-weight (RW)

Annex 1. Simulated Data Analysis.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
#	**n**	**PD**	**r**	**Φ2**	**iter., #**	**error**	**mean_DR**	**Var_DR**	**AVar_Corr Suggested**	**AVar_Corr Wunderer**	**AVar indep**	**Φ2 empir Wunderer**	**r_input (=col. 4)**	**r_empir suggested**	**r_empir Wunderer**
1	10	5%	-10%	0,16%	10000	31,80%	5,40%	0,25%	0,05%	0,39%	0,48%	0,00%	-10%	-6%	-60%
2	10	5%	-5%	0,20%	10000	12,60%	5,40%	0,25%	0,26%	0,43%	0,48%	0,00%	-5%	-6%	-60%
3	10	5%	0	0,25%	10000	38,00%	6,00%	0,38%	0,48%	0,48%	0,48%	0,16%	0%	-4%	-10%
4	10	5%	5%	0,31%	10000	53,90%	4,20%	0,39%	0,69%	0,53%	0,48%	0,16%	5%	0%	-5%
5	10	5%	10%	0,37%	10000	80,10%	4,00%	0,63%	0,90%	0,58%	0,48%	0,43%	10%	7%	25%
6	10	5%	20%	0,52%	10000	86,00%	5,80%	1,42%	1,33%	0,72%	0,48%	1,30%	20%	18%	50%
7	10	10%	-10%	0,72%	10000	22,40%	9,70%	0,13%	0,09%	0,65%	0,90%	0,11%	-10%	-9%	-45%
8	10	10%	-5%	0,85%	10000	23,60%	10,10%	0,45%	0,50%	0,77%	0,90%	0,52%	-5%	-6%	-20%
9	10	10%	0	1,00%	10000	29,60%	9,50%	0,88%	0,90%	0,90%	0,90%	0,92%	0%	0%	0
10	10	10%	5%	1,16%	10000	33,00%	7,70%	1,11%	1,31%	1,04%	0,90%	1,03%	5%	6%	20%
11	10	10%	10%	1,33%	10000	28,40%	9,70%	1,63%	1,71%	1,20%	0,90%	1,77%	10%	10%	25%
12	10	10%	20%	1,72%	10000	61,30%	8,20%	1,83%	2,52%	1,55%	0,90%	1,86%	20%	16%	35%
13	10	25%	-10%	5,26%	10000	18,40%	23,10%	0,40%	0,19%	0,98%	1,88%	3,80%	-10%	-9%	-20%
14	10	25%	-5%	5,75%	10000	12,50%	22,60%	1,00%	1,03%	1,43%	1,88%	4,28%	-5%	-5%	-10%
15	10	25%	0	6,25%	10000	12,30%	24,10%	1,74%	1,88%	1,88%	1,88%	5,71%	0%	-1%	-5%
16	10	25%	5%	6,75%	10000	10,90%	26,80%	2,81%	2,72%	2,33%	1,88%	8,12%	5%	5%	10%
17	10	25%	10%	7,28%	8111	9,80%	27,30%	3,75%	3,56%	2,80%	1,88%	9,42%	10%	10%	15%
18	10	25%	20%	8,37%	10000	13,00%	23,60%	4,94%	5,25%	3,78%	1,88%	9,05%	20%	19%	35%
19	10	50%	-10%	23,41%	10000	11,10%	47,90%	0,47%	0,25%	1,07%	2,50%	20,69%	-10%	-9%	-10%
20	10	50%	-5%	24,20%	4039	9,70%	48,90%	1,39%	1,38%	1,78%	2,50%	22,68%	-5%	-5%	-5%
21	10	50%	0	25,00%	2592	9,40%	48,80%	2,55%	2,50%	2,50%	2,50%	23,87%	0%	0%	5%
22	10	50%	5%	25,80%	3766	9,40%	51,50%	3,62%	3,63%	3,22%	2,50%	27,77%	5%	5%	10%
23	10	50%	10%	26,59%	6128	10,20%	54,10%	4,67%	4,75%	3,93%	2,50%	31,70%	10%	10%	15%
24	10	50%	20%	28,20%	4933	9,10%	49,40%	7,01%	7,00%	5,38%	2,50%	29,41%	20%	20%	30%

Annex 2a. regulatory vs. suggested approach (illustrative case, mean DR=5%).

	Data Descriptives														
1	mean_DR	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%
2	Var_DR	1,0%	1,0%	1,0%	1,0%	1,0%	1,0%	1,0%	2,0%	2,0%	2,0%	2,0%	2,0%	2,0%	2,0%
3	Exposure type	Corporate	SME	HVCRE	QRR	Mortgage	Other retail	SIFI	Corporate	SME	HVCRE	QRR	Mortgage	Other retail	SIFI
4	row # (Annex 1)	1	2	3	4	5	6	7	1	2	3	4	5	6	7
	Capital requirements assuming correlated Gaussian asset returns (K0)														
5	r0 = sqrt (R0)	36,03%	29,98%	36,71%	20,00%	38,73%	22,93%	40,29%	36,03%	29,98%	36,71%	20,00%	38,73%	22,93%	40,29%
	R_0	*n/a*	*-4%*	*n/a*	*4%*	*15%*	*n/a*	*n/a*	*n/a*	*-4%*	*n/a*	*4%*	*15%*	*n/a*	*n/a*
	R_min	*12%*	*12%*	*12%*	*n/a*	*n/a*	*3%*	*15%*	*12%*	*12%*	*12%*	*n/a*	*n/a*	*3%*	*15%*
	R_max	*24%*	*24%*	*30%*	*n/a*	*n/a*	*16%*	*30%*	*24%*	*24%*	*30%*	*n/a*	*n/a*	*16%*	*30%*
	coef	*-50*	*-50*	*-50*	*n/a*	*n/a*	*-35*	*-50*	*-50*	*-50*	*-50*	*n/a*	*n/a*	*-35*	*-50*
6	R0	12,99%	8,99%	13,48%	4,00%	15,00%	5,26%	16,23%	12,99%	8,99%	13,48%	4,00%	15,00%	5,26%	16,23%
	mean_DR	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*
	conf.level	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*
7	K0	28,45%	22,57%	29,16%	14,73%	31,35%	16,81%	33,11%	28,45%	22,57%	29,16%	14,73%	31,35%	16,81%	33,11%
	Capital requirements assuming correlated discrete default events (K1)														
8	r1	21,05%	21,05%	21,05%	21,05%	21,05%	21,05%	21,05%	42,11%	42,11%	42,11%	42,11%	42,11%	42,11%	42,11%
9	R1 = r1 ^ 2	4,43%	4,43%	4,43%	4,43%	4,43%	4,43%	4,43%	17,73%	17,73%	17,73%	17,73%	17,73%	17,73%	17,73%
	mean_DR	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*	*5,0%*
	conf.level	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*
10	K1	15,46%	15,46%	15,46%	15,46%	15,46%	15,46%	15,46%	35,24%	35,24%	35,24%	35,24%	35,24%	35,24%	35,24%
	Comparison of Capital Requirements														
11	K1/K0	54,3%	68,5%	53,0%	104,9%	49,3%	92,0%	46,7%	123,9%	156,1%	120,8%	239,2%	112,4%	209,7%	106,4%
12	CV	20,0%	20,0%	20,0%	20,0%	20,0%	20,0%	20,0%	40,0%	40,0%	40,0%	40,0%	40,0%	40,0%	40,0%

Annex 2b. regulatory vs suggested approach (illustrative case, mean DR=10%).

	Data Descriptives														
1	mean_DR	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%
2	Var_DR	1,0%	1,0%	1,0%	1,0%	1,0%	1,0%	1,0%	2,0%	2,0%	2,0%	2,0%	2,0%	2,0%	2,0%
3	Exposure type	Corporate	SME	HVCRE	QRR	Mortgage	Other retail	SIFI	Corporate	SME	HVCRE	QRR	Mortgage	Other retail	SIFI
4	row # (Annex 1)	1	2	3	4	5	6	7	1	2	3	4	5	6	7
	Capital requirements assuming correlated Gaussian asset returns (K0)														
5	r0 = sqrt (R0)	34,76%	28,43%	34,82%	20,00%	38,73%	18,42%	38,86%	34,76%	28,43%	34,82%	20,00%	38,73%	18,42%	38,86%
	R_0	*n/a*	*-4%*	*n/a*	*4%*	*15%*	*n/a*	*n/a*	*n/a*	*-4%*	*n/a*	*4%*	*15%*	*n/a*	*n/a*
	R_min	*12%*	*12%*	*12%*	*n/a*	*n/a*	*3%*	*15%*	*12%*	*12%*	*12%*	*n/a*	*n/a*	*3%*	*15%*
	R_max	*24%*	*24%*	*30%*	*n/a*	*n/a*	*16%*	*30%*	*24%*	*24%*	*30%*	*n/a*	*n/a*	*16%*	*30%*
	coef	*-50*	*-50*	*-50*	*n/a*	*n/a*	*-35*	*-50*	*-50*	*-50*	*-50*	*n/a*	*n/a*	*-35*	*-50*
6	R0	12,08%	8,08%	12,12%	4,00%	15,00%	3,39%	15,10%	12,08%	8,08%	12,12%	4,00%	15,00%	3,39%	15,10%
	mean_DR	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*
	conf.level	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*
7	K0	41,24%	33,71%	41,32%	24,91%	46,34%	23,43%	46,51%	41,24%	33,71%	41,32%	24,91%	46,34%	23,43%	46,51%
	Capital requirements assuming correlated discrete default events (K1)														
8	r1	11,11%	11,11%	11,11%	11,11%	11,11%	11,11%	11,11%	22,22%	22,22%	22,22%	22,22%	22,22%	22,22%	22,22%
9	R1 = r1 ^ 2	1,23%	1,23%	1,23%	1,23%	1,23%	1,23%	1,23%	4,94%	4,94%	4,94%	4,94%	4,94%	4,94%	4,94%
	mean_DR	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*	*10,0%*
	conf.level	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*
10	K1	17,26%	17,26%	17,26%	17,26%	17,26%	17,26%	17,26%	27,09%	27,09%	27,09%	27,09%	27,09%	27,09%	27,09%
	Comparison of Capital Requirements														
11	K1/K0	41,8%	51,2%	41,8%	69,3%	37,2%	73,7%	37,1%	65,7%	80,4%	65,6%	108,7%	58,5%	115,6%	58,2%
12	CV	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	10,0%	20,0%	20,0%	20,0%	20,0%	20,0%	20,0%	20,0%

Annex 3. regulatory vs suggested approach (empirical data).

#		Moody's		S&P		Moody's		S&P		Moody's		S&P	
	Data Descriptives												
1	Grade Type	**Investement grade (IG)**				**ALL (IG + SG)**				**Speculative grade (SG)**			
2	Year Start	1983	1983	1981	1981	1983	1983	1981	1981	1983	1983	1981	1981
3	Year End	2003	2017	2003	2018	2003	2017	2003	2018	2003	2017	2003	2018
4	mean_DR	0,038%	0,049%	0,112%	0,091%	1,014%	0,963%	1,567%	1,443%	2,948%	2,566%	4,771%	3,994%
5	*change after 2003*		*27,5%*		*-19,0%*		*-5,1%*		*-7,9%*		*-13,0%*		*-16,3%*
6	Var_DR	0,00005%	0,00010%	0,00011%	0,00014%	0,00574%	0,00587%	0,01031%	0,00969%	0,03829%	0,03811%	0,07437%	0,07158%
7	*change after 2003*		*75,9%*		*22,6%*		*2,1%*		*-6,0%*		*-0,5%*		*-3,8%*
	Capital requirements assuming correlated Gaussian asset returns (K0)												
8	r0 = sqrt (R0)	48,76%	48,70%	48,32%	48,44%	43,85%	44,06%	41,81%	42,23%	38,40%	39,15%	36,20%	36,92%
	R_0	*n/a*	*n/a*	*n/a*	*n/a*	*n/a*	*n/a*	*n/a*	*n/a*	*n/a*	*n/a*	*n/a*	*n/a*
	R_min	*12%*	*12%*	*12%*	*12%*	*12%*	*12%*	*12%*	*12%*	*12%*	*12%*	*12%*	*12%*
	R_max	*24%*	*24%*	*24%*	*24%*	*24%*	*24%*	*24%*	*24%*	*24%*	*24%*	*24%*	*24%*
	coef	*-50*	*-50*	*-50*	*-50*	*-50*	*-50*	*-50*	*-50*	*-50*	*-50*	*-50*	*-50*
9	R0	23,77%	23,71%	23,35%	23,47%	19,23%	19,41%	17,48%	17,83%	14,75%	15,33%	13,10%	13,63%
	mean_DR	*0,038%*	*0,049%*	*0,112%*	*0,091%*	*1,014%*	*0,963%*	*1,567%*	*1,443%*	*2,948%*	*2,566%*	*4,771%*	*3,994%*
	conf.level	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*
10	K0	1,66%	2,00%	3,70%	3,18%	14,12%	13,78%	17,17%	16,57%	22,36%	21,09%	27,80%	25,56%
11	*change after 2003*		*20,5%*		*-14,1%*		*-2,5%*		*-3,5%*		*-5,7%*		*-8,1%*
	Capital requirements assuming correlated discrete default events (K1)												
12	r1	0,14%	0,20%	0,10%	0,15%	0,57%	0,62%	0,67%	0,68%	1,34%	1,52%	1,64%	1,87%
13	R1 = r1 ^ 2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,02%	0,02%	0,03%	0,03%
	mean_DR	*0,038%*	*0,049%*	*0,112%*	*0,091%*	*1,014%*	*0,963%*	*1,567%*	*1,443%*	*2,948%*	*2,566%*	*4,771%*	*3,994%*
	conf.level	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*	*99,9%*
14	K1	0,04%	0,05%	0,11%	0,09%	1,06%	1,01%	1,65%	1,52%	3,23%	2,86%	5,29%	4,51%
15	*change after 2003*		*28,2%*		*-18,5%*		*-4,7%*		*-7,8%*		*-11,6%*		*-14,7%*
	Comparison of Capital Requirements												
16	K1/K0	2,3%	2,5%	3,0%	2,9%	7,5%	7,4%	9,6%	9,2%	14,5%	13,6%	19,0%	17,7%
17	CV	0,1%	0,2%	0,1%	0,2%	0,6%	0,6%	0,7%	0,7%	1,3%	1,5%	1,6%	1,8%

Annex 4. Values of bivariate joint standard normal cumulative density function (Φ2 or $N(\gamma,\gamma,R)$ where $\gamma = N^{-1}(PD)$, $R = Corr(Z_i; Z_j)$), in percentage points (pp) out of 100.

		R, %																
		-100	-75	-50	-25	-20	-15	-10	-5	0	+5	+10	+15	+20	+25	+50	+75	+100
PD, %	1															0,1	0,3	1,0
	5				0,1	0,1	0,1	0,2	0,2	0,3	0,3	0,4	0,4	0,5	0,6	1,2	2,2	5,0
	10			0,1	0,4	0,5	0,6	0,7	0,9	1,0	1,2	1,3	1,5	1,7	1,9	3,2	5,1	10,0
	20		0,1	0,8	2,2	2,5	2,9	3,2	3,6	4,0	4,4	4,8	5,2	5,7	6,1	8,7	12,1	20,0
	25		0,3	1,8	3,9	4,3	4,8	5,3	5,8	6,3	6,8	7,3	7,8	8,4	8,9	12,0	15,9	25,0
	30		0,9	3,3	6,1	6,6	7,2	7,8	8,4	9,0	9,6	10,2	10,9	11,5	12,1	15,7	20,0	30,0
	40		4,1	8,4	12,3	13,0	13,8	14,5	15,3	16,0	16,7	17,5	18,3	19,0	19,8	23,9	28,9	40,0
	50		11,5	16,7	21,0	21,8	22,6	23,4	24,2	25,0	25,8	26,6	27,4	28,2	29,0	33,3	38,5	50,0
	60	20,0	24,1	28,4	32,3	33,0	33,8	34,5	35,3	36,0	36,7	37,5	38,3	39,0	39,8	43,9	48,9	60,0
	70	40,0	40,9	43,3	46,1	46,6	47,2	47,8	48,4	49,0	49,6	50,2	50,9	51,5	52,1	55,7	60,0	70,0
	80	60,0	60,1	60,8	62,2	62,5	62,9	63,2	63,6	64,0	64,4	64,8	65,2	65,7	66,1	68,7	72,1	80,0
	90	80,0	80,0	80,1	80,4	80,5	80,6	80,7	80,9	81,0	81,2	81,3	81,5	81,7	81,9	83,2	85,1	90,0
	99	98,0	98,0	98,0	98,0	98,0	98,0	98,0	98,0	98,0	98,0	98,0	98,0	98,0	98,0	98,1	98,3	99,0

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